

On the geodetic and the hull numbers in strong product graphs[☆]

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Abstract

A set S of vertices of a connected graph G is convex, if for any pair of vertices $u, v \in S$, every shortest path joining u and v is contained in S . The convex hull $CH(S)$ of a set of vertices S is defined as the smallest convex set in G containing S . The set S is geodetic, if every vertex of G lies on some shortest path joining two vertices in S , and it is said to be a hull set if its convex hull is $V(G)$. The geodetic and the hull numbers of G are the cardinality of a minimum geodetic and a minimum hull set, respectively. In this work, we investigate the behavior of both geodetic and hull sets with respect to the strong product operation for graphs. We also establish some bounds for the geodetic number and the hull number and obtain the exact value of these parameters for a number of strong product graphs.

Keywords: Metric graph theory, Geodetic set, Hull set, Geodetic number, Hull number, Strong product

1. Introduction

The process of rebuilding a network modelled by a connected graph is a discrete optimization problem, consisting in finding a subset of vertices of cardinality as small as possible, which, roughly speaking, would allow us to store and retrieve the whole graph. One way to approach this problem is by using a certain convex operator. This procedure has attracted much attention since it was shown by Farber and Jamison [13] that every convex subset in a graph is the convex hull of

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its extreme vertices if and only if the graph is chordal and contains no induced 3-fan. From then on, a number of variants of this approach have been proposed [5, 22]. One of them, consists in using, instead of the convex hull operator, the closed interval operator, i.e., considering geodetic sets instead of hull sets [3, 4]. Unfortunately, computing geodetic sets and hull sets of minimum cardinality, are known to be NP-hard problems for general graphs [10, 11]. This fact has motivated the study of these two problems for graph classes which can be obtained by means of graph operations, such as cartesian product [1, 6, 17], composition [7] and join [8]. Let us notice that in these graphs, information about factor graphs can be used to obtain geodetic and hull sets and to compute geodetic and hull numbers.

In this work, we study both geodetic and hull sets of minimum cardinality, in strong product graphs. This graph operation has been extensively investigated in relation to a wide range of subjects, including: connectivity [2], pancyclicity [19, 20], chromaticity [25], bandwidth [18], independency [14, 23] and primitivity [21]. Section 2 is devoted to introduce the main definitions and notation used throughout the paper. In Section 3, we study the behavior of geodetic and hull sets with respect to the strong product operation. In Section 4, a number of lower and upper sharp bounds for the geodetic number and the hull number of the strong product of two graphs are presented. Finally, the last section is devoted to obtain the exact value of the geodetic number and the hull number of the strong product of some basic families of graphs, such as paths, complete graphs and cycles.

2. Graph theoretical preliminaries

We consider only finite, simple, connected graphs. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [24]. Given vertices u, v in a graph G we let $d_G(u, v)$ denote the distance between u and v in G . When there is no confusion, subscripts will be omitted. The diameter $diam(G)$ of G is the maximum distance between any two vertices of G . An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. The *closed interval* $I[x, y]$ consists of x, y and all vertices lying in some $x - y$ geodesic of G . For $S \subseteq V(G)$, the *geodetic closure* $I[S]$ of S is the union of all closed intervals $I[u, v]$ over all pairs $u, v \in S$, i.e., $I[S] = \bigcup_{u, v \in S} I[u, v]$. The set S is called *geodetic* if $I[S] = V(G)$ and it is said to be *convex* if $I[S] = S$. The *convex hull*

$CH(S)$ of S is the smallest convex set containing S . If we define $I^0[S] = S$, $I^i[S] = I[I^{i-1}[S]$ for every $i \geq 1$, then $CH(S) = I^r[S]$, for some $r \geq 0$. The set S is said to be a *hull set* if its convex hull $CH(S)$ is the whole vertex set $V(G)$. The *geodetic number* $g(G)$ and the *hull number* $h(G)$ are the minimum cardinality of a geodetic set and a hull set, respectively [12, 15]. Certainly, every geodetic set is a hull set, and hence, $h(G) \leq g(G)$. In Table 1, both the geodetic number and the hull number of some families of graphs are shown.

| G | P_n | C_{2l} | C_{2l+1} | T_n^h | K_n | $K_{p,n-p}$ | $S_{1,n-1}$ | $W_{1,n-1}$ |
|--------|-------|----------|------------|---------|-------|----------------|-------------|-------------------------------|
| $h(G)$ | 2 | 2 | 3 | h | n | 2 | $n-1$ | $\lceil \frac{n-1}{2} \rceil$ |
| $g(G)$ | 2 | 2 | 3 | h | n | $\min\{4, p\}$ | $n-1$ | $\lceil \frac{n-1}{2} \rceil$ |

Table 1: Hull number and geodetic number of some graph classes.

Remark 1. In the rest of this paper, P_n , C_n and K_n denote the path, cycle and complete graph of order n , respectively. In all cases, unless otherwise stated, the set of vertices is $\{0, 1, \dots, n-1\}$. In addition, $K_{p,n-p}$, $S_{1,n-1}$, $W_{1,n-1}$ denote the complete bipartite graph (being its smallest stable set of order $p \geq 2$), star and wheel of order n , whereas T_n^h represents an arbitrary tree of order n with h leaves. Finally, in the sequel, G and H denote a pair of nontrivial connected graphs.

A vertex $v \in V(G)$ is a *simplicial vertex* if the subgraph induced by its neighborhood $N(v) = \{u : uv \in E(G)\}$ is a complete graph. It is easily seen that every hull set, and hence every geodetic set, must contain the set $Ext(G)$ of simplicial vertices of G . A graph G is called *extreme geodesic* if the set of its simplicial vertices is geodetic (see [9]). Note that, in this case, (1) the set $Ext(G)$ is the unique minimum geodetic set (and also the unique minimum hull set) and (2) $h(G) = g(G) = |Ext(G)|$. Trees and complete graphs are basic examples of extreme geodesic graphs.

3. Strong product of graphs: general results

The *strong product* of graphs G and H , denoted by $G \boxtimes H$, is the graph with the vertex set $V(G) \times V(H) = \{(g, h) : g \in V(G), h \in V(H)\}$ in which vertices (g, h) and (g', h') are adjacent whenever (1) $g = g'$ and $hh' \in E(H)$, or (2) $h = h'$ and $gg' \in E(G)$, or (3) $gg' \in E(G)$ and $hh' \in E(H)$.

The most important metric property of the strong product operation, relating the distance between two arbitrary vertices of an strong product graph to the distances between the corresponding projections in its factors, is shown next.

Lemma 1. ([16]) *If $(g, h), (g', h') \in V(G \boxtimes H)$, then $d_{G \boxtimes H}((g, h), (g', h')) = \max\{d_G(g, g'), d_H(h, h')\}$. Hence, $\text{diam}(G \boxtimes H) = \max\{\text{diam}(G), \text{diam}(H)\}$.*

In this section, we firstly present some lemmas in order to show the behavior of the closed interval operator with respect to the strong graph operation, and next, we analyze in which way, both geodetic and hull sets of the strong product of two graphs, are related to geodetic and hull sets of each factor, in both directions.

In the sequel, $p_G(S)$ and $p_H(S)$ denote the projections of a set of vertices $S \subseteq V(G \boxtimes H)$ onto G and H , respectively.

Lemma 2. *Let $u = (g, h), v = (g', h') \in V(G \boxtimes H)$ such that $d_{G \boxtimes H}(u, v) = d_G(g, g') = l$. If γ is a $(g, h) - (g', h')$ geodesic, then the projection of γ onto G is a $g - g'$ geodesic of length l .*

Proof. If $V(\gamma) = \{(g, h), (g_1, h_1), \dots, (g_{l-1}, h_{l-1}), ((g', h'))\}$, then its projection into G is $p_G(V(\gamma)) = \{g, g_1, \dots, g_{l-1}, g'\}$. Since $d_{G \boxtimes H}((g, h), (g', h')) = d_G(g, g')$, $p_G(V(\gamma))$ does not contain repeated vertices, which means that every pair of consecutive vertices are adjacent, i.e., $p_G(V(\gamma))$ is the vertex set of a $g - g'$ geodesic in G . \square

Lemma 3. *Let $u = (g_1, h_1), v = (g_2, h_2) \in V(G \boxtimes H)$ such that $d_{G \boxtimes H}(u, v) = d_G(g_1, g_2) = l$. Then,*

$$I[u, v] = \{(g, h) : g \in I[g_1, g_2], d_H(h_1, h) \leq d_G(g_1, g), d_H(h, h_2) \leq d_G(g, g_2)\}.$$

Proof. Let $w = (g, h)$ be a vertex belonging to $I[u, v]$. By Lemma 2, the projection of every $u - v$ geodesic onto G is a $g_1 - g_2$ geodesic, which means that $g \in I[g_1, g_2]$. If $d_G(g_1, g) > d_H(h_1, h)$, then

$$\begin{aligned} d_G(g_1, g_2) &= d_{G \boxtimes H}(u, v) = d_{G \boxtimes H}(u, w) + d_{G \boxtimes H}(w, v) \\ &= d_H(h_1, h) + \max\{d_G(g, g_2), d_H(h, h_2)\} \\ &> d_G(g_1, g) + d_G(g, g_2) = d_G(g_1, g_2), \end{aligned}$$

which is a contradiction. Similarly, a contradiction is obtained by assuming that $d_H(h, h_2) > d_G(g, g_2)$.

Conversely, suppose that $w = (g, h)$ is a vertex belonging to $V(G \boxtimes H)$ such that $g \in I[g_1, g_2]$, $r = d_H(h_1, h) \leq d_G(g_1, g) = k$, $s = d_H(h, h_2) \leq d_G(g, g_2) = l - k$. Let ρ be a $g_1 - g_2$ geodesic passing through g such that $V(\rho) = \{z_0, z_1, \dots, z_l\}$, $z_0 = g_1$, $z_k = g$ and $z_l = g_2$. Let μ_1 be a $h_1 - h$ geodesic such that $V(\mu_1) = \{x_0, x_1, \dots, x_r\}$, $x_0 = h_1$ and $x_r = h$. Let μ_2 be a $h - h_2$ geodesic such that $V(\mu_2) = \{y_0, y_1, \dots, y_s\}$, $y_0 = h$ and $y_s = h_2$. It is straightforward to check that

$$\{(z_0, x_0), (z_1, x_1), \dots, (z_r, x_r), \dots, (z_k, x_r), (z_{k+1}, y_1), \dots, (z_{k+s}, y_s), \dots, (z_l, y_s)\}$$

is the vertex set of a $u - v$ geodesic passing through w , which means that $w \in I[u, v]$. \square

Lemma 4. Let $S_1 \times S_2 \subseteq V(G \boxtimes H)$ a set of vertices of cardinality 6, where $S_1 = \{g_1, g_2\} \subseteq V(G)$ and $S_2 = \{h_1, h_2, h_3\} \subseteq V(H)$. Then

- (i) $(g_2, h_2) \notin I[(g_1, h_1), (g_1, h_2), (g_2, h_1)]$
- (ii) If $h_3 \notin I[h_1, h_2]$, then $(g_2, h_3) \notin I[(g_1, h_1), (g_1, h_2)]$.
- (iii) If $h_3 \notin I[h_1, h_2]$, then $(g_1, h_3) \notin I[(g_1, h_1), (g_2, h_2)]$.

Proof. (i) Observe that $d((g_1, h_1), (g_1, h_2)) = d(h_1, h_2)$. Hence, according to Lemma 2, every $(g_1, h_1) - (g_1, h_2)$ geodesic may not pass through (g_2, h_2) . Similarly, it is proved that $(g_2, h_2) \notin I[(g_1, h_1), (g_2, h_1)]$ and $(g_2, h_2) \notin I[(g_1, h_2), (g_2, h_1)]$ (see Figure 1(a)).

(ii) Observe that $d((g_1, h_1), (g_1, h_2)) = d(h_1, h_2)$. Hence, according to Lemma 2, the projection onto H of a $(g_1, h_1) - (g_1, h_2)$ geodesic passing through (g_2, h_3) is a $h_1 - h_2$ geodesic passing through h_3 , contradicting the hypothesis $h_3 \notin I[h_1, h_2]$ (see Figure 1(b)).

(iii) Suppose that $(g_1, h_3) \in I[(g_1, h_1), (g_2, h_2)]$.

If $d((g_1, h_1), (g_2, h_2)) = d(g_1, g_2)$, then, according to Lemma 2, every $(g_1, h_1) - (g_2, h_2)$ geodesic may not pass through (g_1, h_3) .

If $d((g_1, h_1), (g_2, h_2)) = d(h_1, h_2)$, then the projection onto H of a $(g_1, h_1) - (g_2, h_2)$ geodesic passing through (g_1, h_3) is a $h_1 - h_2$ geodesic passing through h_3 , which contradicts the hypothesis $h_3 \notin I[h_1, h_2]$ (see Figure 1(c)). \square

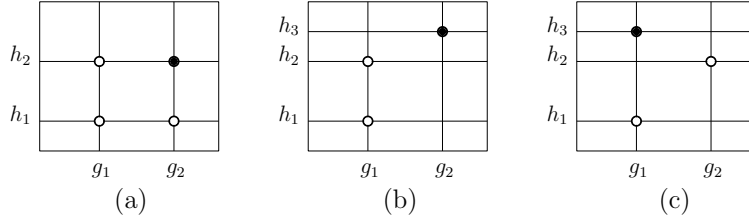


Figure 1: In each figure, the black vertex does not belong to the geodetic closure of white vertices

Lemma 5. Let $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$. Then, for every integer $r \geq 1$, $I^r[S_1] \times I^r[S_2] \subseteq I^r[S_1 \times S_2]$.

Proof. We proceed by induction on r . Suppose that $r = 1$ and take a vertex $(g, h) \in I[S_1] \times I[S_2]$. Since $g \in I[S_1]$, then $g \in I[g', g'']$ for some $g', g'' \in V(S_1)$, and thus $d(g', g'') = d(g', g) + d(g, g'')$. Similarly, $d(h', h'') = d(h', h) + d(h, h'')$ for some $h', h'' \in V(S_2)$. We may assume without loss of generality that $d(g', g) \leq d(g, g'')$, $d(h', h) \leq d(h, h'')$ and $d(g', g) \leq d(h', h)$. Then, $d((g', h'), (g, h)) = d(h', h)$ and $d((g, h), (g', h'')) = d(h, h'')$, which means that

$$d((g', h'), (g', h'')) = d(h', h'') = d(h', h) + d(h, h'') = d((g', h'), (g, h)) + d((g, h), (g', h''))$$

In other words, $(g, h) \in I[(g', h'), (g', h'')] \subseteq I[S_1 \times S_2]$.

Assume then that $r > 1$. By the inductive hypothesis, $I^{r-1}[S_1] \times I^{r-1}[S_2] \subseteq I^{r-1}[S_1 \times S_2]$. Hence, $I^r[S_1] \times I^r[S_2] = I[I^{r-1}[S_1]] \times I[I^{r-1}[S_2]] \subseteq I[I^{r-1}[S_1] \times I^{r-1}[S_2]] \subseteq I[I^{r-1}[S_1 \times S_2]] = I^r[S_1 \times S_2]$. \square

As a direct consequence of this lemma, the following result is obtained.

Proposition 1. Let $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$. If S_1 is a geodetic (resp. hull) set of G and S_2 is a geodetic (resp. hull) set of H , then $S_1 \times S_2$ is a geodetic (resp. hull) set of $G \boxtimes H$.

Proof. Let r, s be positive integers such that $I^r[S_1] = V(G)$ and $I^s[S_2] = V(H)$. We may suppose wlog that $r \leq s$. Then, $V(G \boxtimes H) = V(G) \times V(H) = I^s[S_1] \times I^s[S_2] \subseteq I^s[S_1 \times S_2]$. \square

Proposition 2. Let $S \subseteq V(G \boxtimes H)$ be a geodetic set of $G \boxtimes H$. Then, either the projection of S onto G or the projection of S onto H is geodetic.

Proof. Assume that neither $S_1 = p_G(S)$ nor $S_2 = p_H(S)$ is geodetic and consider $g \in V(G) \setminus I[S_1]$ and $h \in V(H) \setminus I[S_2]$. As $(g, h) \in I[S] = V(G \boxtimes H)$, then $(g, h) \in I[(g', h'), (g'', h'')] for some $(g', h'), (g'', h'') \in S$. Hence, $d((g', h'), (g'', h'')) = d((g', h'), (g, h)) + d((g, h), (g'', h''))$.$

On the other hand, as $g \notin I[g', g'']$ and $h \notin I[h', h'']$, we have that $d(g', g'') < d(g', g) + d(g, g'')$ and $d(h', h'') < d(h', h) + d(h, h'')$. Hence,

$$\begin{aligned} \max\{d(g', g''), d(h', h'')\} &< \max\{d(g', g) + d(g, g''), d(h', h) + d(h, h'')\} \leq \\ &\leq \max\{d(g', g), d(h', h)\} + \max\{d(g, g''), d(h, h'')\} = d((g', h'), (g, h)) + d((g, h), (g'', h'')) \end{aligned}$$

which contradicts the previous expression for the distance between (g', h') and (g'', h'') . \square

This property is far from being true for hull sets, as it is shown in the next example.

Example 1. It is straightforward to prove that if $V(C_5) = \{u_1, \dots, u_5\}$ and $V(C_7) = \{v_1, \dots, v_7\}$, then (1) $\{(u_1, u_2)\}$ is not a hull set of C_5 , (2) $\{(v_1, v_4)\}$ is not a hull set of C_7 , and (3) $\{(u_1, v_1), (u_2, v_4)\}$ is a hull set of $C_5 \boxtimes C_7$.

4. Geodetic and hull numbers: bounds

In this section, we study the behavior of both the geodetic and the hull numbers with respect to the strong product operation for graphs, in terms of its factors. More precisely, we obtain bounds, and we give some examples showing that all of them are sharp.

Lemma 6. Let $\{h_1, h_2, h_3\}$ a 3-vertex set of a graph H . If $h_1 \in I[h_2, h_3]$, then $h_2 \notin I[h_1, h_3]$ and $h_3 \notin I[h_1, h_2]$.

Proof. Assume on the contrary that, for example, $h_2 \in I[h_1, h_3]$. Then, if $d(h_1, h_2) = x$, $d(h_1, h_3) = y$ and $d(h_2, h_3) = z$, we have that $x + y = z$ and $x + z = y$, i.e., $d(h_1, h_2) = 0$, a contradiction. \square

Proposition 3. Let G and H be nontrivial graphs. Then, $g(G \boxtimes H) \geq 4$.

Proof. Let us see that every subset S of $V(G \boxtimes H)$ having at most 3 vertices is not geodetic. Suppose on the contrary that S is a geodetic set of cardinality 3. Without loss of generality, we may assume that $|p_G(S)| \leq |p_H(S)|$. We consider different cases.

Case 1. $|p_G(S)| = 1$: In other words, $S = \{(g_1, h_1), (g_1, h_2), (g_1, h_3)\}$ and $|p_H(S)| = 3$. According to Lemma 6, we may assume w.o.l.g. that $h_3 \notin I[h_1, h_2]$ and from Lemma 4(i,ii), we derive that $(g_2, h_3) \notin I[S]$ for any vertex $g_2 \neq g_1$ (see Figure 2(a)).

Case 2. $|p_G(S)| = |p_H(S)| = 2$: In other words, $S = \{(g_1, h_1), (g_1, h_2), (g_2, h_1)\}$, being $g_1 \neq g_2$, and $h_1 \neq h_2$. From Lemma 4(i), we derive that $(g_2, h_2) \notin I[S]$ (see Figure 2(b)).

Case 3. $|p_G(S)| = 2$ and $|p_H(S)| = 3$: In other words, $S = \{(g_1, h_1), (g_1, h_2), (g_2, h_3)\}$, being $g_1 \neq g_2$, and h_1, h_2, h_3 three different vertices of H . According to Lemma 6, we may assume w.o.l.g. that $h_1 \notin I[h_2, h_3]$. From Lemma 4(i,iii), we derive that $(g_2, h_1) \notin I[S]$ (see Figure 2(c)).

Case 4. $|p_G(S)| = |p_H(S)| = 3$: In other words, $S = \{(g_1, h_1), (g_2, h_2), (g_3, h_3)\}$, being g_1, g_2, g_3 three different vertices of G and h_1, h_2, h_3 three different vertices of H . According to Lemma 6, we may assume w.o.l.g. that $h_1 \notin I[h_2, h_3]$ and $g_3 \notin I[g_1, g_2]$. From Lemma 4(i,iii), we derive that $(g_3, h_1) \notin I[S]$ (see Figure 2(d)). \square

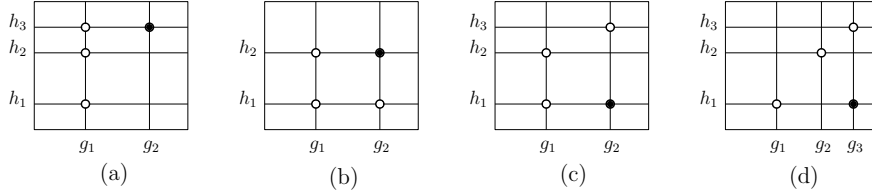


Figure 2: In each figure, the black vertex does not belong to the geodetic closure of white vertices

As a direct consequence of Propositions 1 and 2, we derive bounds for the geodetic number of the strong product of two graphs, in terms of the geodetic numbers of its factor graphs.

Theorem 1. *For any two graphs G and H , $\min\{g(G), g(H)\} \leq g(G \boxtimes H) \leq g(G)g(H)$.*

Furthermore, both bounds are sharp.

Proof. First, we prove the upper bound. Let S_1 and S_2 be geodetic sets of G and H with minimum cardinality, that is, such that $|S_1| = g(G)$ and $|S_2| = g(H)$. By Proposition 1, $S_1 \times S_2$ is a geodetic set of $G \boxtimes H$ with cardinality $|S_1 \times S_2| = |S_1||S_2| = g(S_1)g(S_2)$. Hence, $g(G \boxtimes H) \leq g(G)g(H)$.

To prove the lower bound, take a minimum geodetic set S of $G \boxtimes H$. According to Proposition 2, we may suppose, without loss of generality, that $p_G(S)$ is a geodetic set of G . Hence: $\min\{g(G), g(H)\} \leq g(G) \leq |p_G(S)| \leq |S| = g(G \boxtimes H)$.

To show the sharpness of the upper bound, take $G = K_m$ and $H = K_n$. Then, $g(K_m \boxtimes K_n) = g(K_{mn}) = mn = g(K_m)g(K_n)$. Finally, to show the sharpness of the lower bound, take $G = K_{r,s}$ a complete bipartite graph and $H = K_n$, with $r, s, n \geq 4$. Then, as it will be shown in the next section (see Example 2), $g(K_{r,s} \boxtimes K_n) = 4 = \min\{g(K_{r,s}), g(K_n)\}$. \square

Theorem 2. *For any two nontrivial graphs G and H , $2 \leq h(G \boxtimes H) \leq h(G)h(H)$. Furthermore, both bounds are sharp.*

Proof. First, we prove the upper bound. Let S_1 and S_2 be hull sets of G and H with minimum cardinality, that is, such that $|S_1| = h(G)$ and $|S_2| = h(H)$. By Proposition 1, $S_1 \times S_2$ is a hull set of $G \boxtimes H$ with cardinality $|S_1 \times S_2| = |S_1||S_2| = h(S_1)h(S_2)$. Hence, $h(G \boxtimes H) \leq h(G)h(H)$.

To prove the sharpness of this bound, take $G = K_m$ and $H = K_n$ and notice that $h(K_m \boxtimes K_n) = h(K_{mn}) = mn = h(K_m)h(K_n)$.

Finally, the lower bound is a direct consequence of the fact that $h(G) = 1$ if and only if $G = K_1$. As for its sharpness, it is straightforward to check that $\{(0,0), (0,2)\}$ is a hull set of $P_2 \boxtimes C_4$. \square

Remark 2. Conversely to the geodetic case, the claim $\min\{h(G), h(H)\} \leq h(G \boxtimes H)$ is far from being true in general. A simple counterexample is shown in Example 1.

Lemma 7. *Let G and H be two graphs such that $\text{Ext}(G) = \emptyset$. If S is a hull set of G and x is an arbitrary vertex of H , then $S \times \{x\}$ is a hull set of $G \boxtimes H$.*

Proof. We prove by induction on $m \geq 0$ that for every vertex $h \in V(H)$ such that $d(x, h) = m \geq 0$, if $g \in V(G)$, then the vertex (g, h) is in the convex hull of $S \times \{x\}$.

For $m = 0$, the condition $d(x, h) = m \geq 0$ implies $h = x$. Since S is a hull set of G , for every $g \in V(G)$ we have $g \in I^r[S]$ for some $r \geq 0$. By lemma 5, $(g, x) \in I^r[S] \times I^r[\{x\}] = I^r[S \times \{x\}]$, and consequently, (g, x) is in the convex hull of $S \times \{x\}$.

Suppose now $m > 0$ and consider a vertex $h \in V(H)$ with $d(x, h) = m > 0$. Take a vertex $h' \in V(H)$ such that $d(h, h') = 1$ and $d(h', x) = m - 1$. Since G has no simplicial vertices, for every vertex $g \in V(G)$ there exist vertices g_1, g_2 in G adjacent to g such that $d(g_1, g_2) = 2$. Thus

$d((g_1, h'), (g_2, h')) = 2$, $d((g, h), (g_1, h')) = 1$ and $d((g, h), (g_2, h')) = 1$, that is, (g, h) is in a geodesic between (g_1, h') and (g_2, h') . By inductive hypothesis, (g_1, h') and (g_2, h') are in the convex hull of $S \times \{x\}$. Therefore, (g, h) is in the convex hull of $S \times \{x\}$. \square

As a consequence of the preceding lemma we obtain the following upper bound for the hull number of the strong product of two graphs, if at least one of them has no simplicial vertices.

Theorem 3. *Let G and H be two graphs such that $\text{Ext}(G) = \emptyset$. Then, $h(G \boxtimes H) \leq h(G)$*

Certainly, this last bound is also sharp. Consider, for example the strong product graph $C_m \boxtimes C_n$, being both m and n even. As it will be shown in the next section (see Proposition 11), $h(C_m \boxtimes C_n) = 2 = h(C_m)$.

5. Exact values

In this section, we approach the calculation of the geodetic and the hull numbers of some strong product graphs, where at least one of the factors is either a complete graph or a cycle or a path. We begin by showing a result involving extreme geodesic graphs.

Proposition 4. *Two graphs G and H are extreme geodesic if and only if $G \boxtimes H$ is an extreme geodesic graph.*

Proof. Observe that a vertex (g, h) is a simplicial vertex of $G \boxtimes H$ if and only if both g and h are simplicial vertices of G and H , respectively, i.e., $\text{Ext}(G \boxtimes H) = \text{Ext}(G) \times \text{Ext}(H)$. As a direct consequence of this equality and Proposition 1, we have that two graphs G and H are extreme geodesic if and only if $G \boxtimes H$ is extreme geodesic. \square

Corollary 1. *If both G and H are extreme geodesic graphs, then $h(G \boxtimes H) = g(G \boxtimes H) = g(G)g(H) = h(G)h(H)$.*

As a direct consequence of Corollary 1, the results shown in Table 2 are obtained.

Certainly, cycles are graphs without simplicial vertices, and hence they are not extreme geodesic graphs. This means that the calculation of the geodetic and the hull numbers of strong product graphs of the form $G \boxtimes C_n$, requires a different approach to the previous one. The rest of this section is devoted to this issue.

| G/H | P_n | T_n^k | K_n |
|---------|-------|---------|-------|
| P_m | 4 | $2k$ | $2n$ |
| T_m^h | $2h$ | hk | hn |
| K_m | $2m$ | mk | mn |

Table 2: Hull and geodetic numbers of the strong product of some extreme geodesic graphs.

Definition 1. Let S be a set of vertices in a graph G . Then, S is said to satisfy condition

(A) if, for every vertex $x \in S$, there exist two vertices $y, z \in S - x$ such that $x \in I[y, z]$.

(B) if there are two vertices $x, y \in S$ such that $x \notin I[S - x]$ and $y \notin I[S - y]$.

Lemma 8. Let G be a graph having a geodetic set S satisfying the condition (A). Then, for every vertex $k \in V(K_n)$, $S \times \{k\}$ is a geodetic set of $g(G \boxtimes K_n)$.

Proof. Take an arbitrary vertex $(g, h) \in V(G \boxtimes K_n)$. This means that there exists a pair of vertices $s, s' \in S \setminus \{g\}$ such that $g \in I[s, s']$. Hence, $d((s, k), (s', k)) = d(s, s') = d(s, g) + d(g, s') = d((s, k), (g, h)) + d((g, h), (s', k))$, i.e., $(g, h) \in I[(s, k), (s', k)] \subset I[S]$, as desired. \square

Proposition 5. Let G be a graph with a minimum geodetic set S satisfying condition (A). Then, for every positive integer n , $g(G \boxtimes K_n) = g(G)$.

Proof. As a corollary of Lemma 8 we have that $g(G \boxtimes K_n) \leq g(G)$. To get the equality, suppose that there exists a geodetic set $R = \{(g_1, k_1), (g_2, k_2), \dots, (g_m, k_m)\}$ in $G \boxtimes K_n$ such that $m = |R| < |S| = g(G)$. Consider the set $R' = \{(g_1, k_1), (g_2, k_1), \dots, (g_m, k_1)\}$. For every vertex $(g, k) \in G \boxtimes K_n$ we have that $(g, h) \in I[(g_i, k_i), (g_j, k_j)]$ for some $i, j \in \{1, \dots, m\}$. Hence, $g \neq g_i \neq g_j \neq g$ and $d((g_i, k_1), (g_j, k_1)) = d(g_i, g_j) = d((g_i, k_i), (g_j, k_j)) = d((g_i, k_i), (g, k)) + d((g, k), (g_j, k_j)) = d(g_i, g) + d(g, g_j) = d((g_i, k_1), (g, k)) + d((g, k), (g_j, k_1))$. In other words, $(g, k) \in I[(g_i, k_1), (g_j, k_1)] \subseteq I[R']$. We have thus proved that R' is also a geodetic set of $G \boxtimes K_n$. Furthermore, as a direct consequence of Proposition 2, we conclude that the projection $p_G(R')$ is a geodetic set of G , from which it follows that $|p_G(R')| = |R'| \leq |R| < |S| = g(G)$, a contradiction. \square

Example 2. Consider the complete bipartite graph $K_{r,s}$, with $2 \leq r \leq s$. Notice that if $V(K_{r,s}) = \{u_1, \dots, u_r\} \cup \{v_1, \dots, v_s\}$, then the set $\{u_1, u_2, v_1, v_2\}$ is a minimum geodetic set satisfying condition (A). Hence, $g(K_{r,s} \boxtimes K_n) = g(K_{r,s}) = 4$.

Proposition 6. Let $n \geq 4$ be an even integer and let G be a graph of order $m \geq 2$. If G is either a path P_m or a complete graph K_m , then: $g(G \boxtimes C_n) = 4$ and $h(G \boxtimes C_n) = 2$.

Proof. The equality $h(G \boxtimes C_n) = 2$ is a direct consequence of Theorem 3. The equality $g(P_m \boxtimes C_n) = 4$ is a corollary of Proposition 3 and the upper bound shown in Theorem 1. Finally, to prove that $g(K_m \boxtimes C_n) = 4$ it is enough to consider again Proposition 3 and to notice that the set $S = \{0, 1, \frac{n}{2}, \frac{n+2}{2}\}$ is a (not minimum) geodetic set in C_n satisfying condition (A). \square

Lemma 9. Let $h \geq 2$ be an integer and let S be a set of vertices in the cycle C_{2h+1} . If $2 \leq |S| \leq 4$, then S satisfies condition (B).

Proof. Certainly, this statement is trivial if $|S| = 2$. For $|S| = 3$, we may assume that $S = \{0, i, j\}$, where $0 < i < j \leq 2h$. If $i > h$, then $0 \notin I[i, j]$ and $i \notin I[0, j]$. If $j \leq h$, then $0 \notin I[i, j]$ and $j \notin I[0, i]$. If $i \leq h$ and $j \geq h + 1$, then $i \notin I[0, j]$ and $j \notin I[0, i]$.

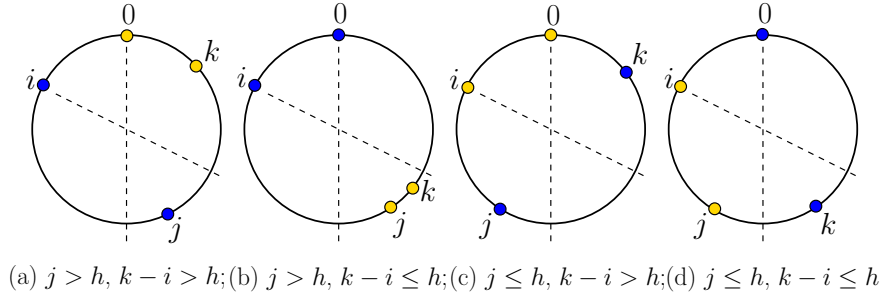


Figure 3: In all cases, $S = \{0, i, j, k\}$ and $0 < i < j < k \leq 2h$.

For $|S| = 4$, we may assume that $S = \{0, i, j, k\}$, where $0 < i < j < k \leq 2h$. If $j > h$ and $k - i > h$, then $i \notin I[S - i]$ and $j \notin I[S - j]$. If $j > h$ and $k - i \leq h$, then $i \notin I[S - i]$ and $0 \notin I[S - 0]$. If $j \leq h$ and $k - i > h$, then $k \notin I[S - k]$ and $j \notin I[S - j]$. If $j \leq h$ and $k - i \leq h$, then $k \notin I[S - k]$ and $0 \notin I[S - 0]$ (see Figure 3). \square

Proposition 7. Let G be a nontrivial graph such that every set of vertices $S \subseteq V(G)$ of cardinality $2 \leq |S| \leq 4$ satisfies condition (B). Then, for every integer $k \geq 2$, $g(G \boxtimes C_{2k+1}) \geq 5$.

Proof. Denote $H = C_{2k+1}$ and assume that $g(G \boxtimes H) = 4$, i.e., that S is a geodetic set of cardinality 4. Observe that $1 \leq |p_G(S)| \leq 4$ and $1 \leq |p_H(S)| \leq 4$. We distinguish three cases.

Case 1. $|p_G(S)| = 1$ or $|p_H(S)| = 1$: If $|p_G(S)| = 1$, as H satisfies condition (B), there exists a vertex $h \in p_H(S)$ s.t. $h \notin I[p_H(S) - h]$. If $p_G(S) = \{g_1\}$ and $g_1g \in E(G)$ then, according to Lemma 4, $(g, h) \notin I[S]$ (see Figure 4(a)). The case $|p_H(S)| = 1$ is similarly proved (see Figure 4(b)).

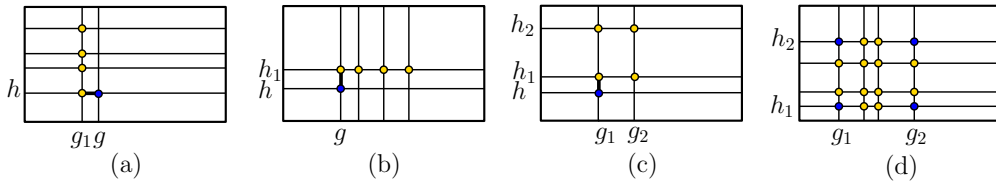


Figure 4: In all cases, each dark vertex is not in the geodetic closure of the remaining vertices.

Case 2. $|p_G(S)| = |p_H(S)| = 2$: If $p_G(S) = \{g_1, g_2\}$ and $p_H(S) = \{h_1, h_2\}$. If h is the vertex adjacent to h_1 not belonging to $I[h_1, h_2]$ then, by Lemma 4, $(g_1, h) \notin I[S]$ (see Figure 4(c)).

Case 3. $|p_G(S)| \cdot |p_H(S)| > 4$: As both G and H satisfy condition (B), there exist vertices g_1, g_2 in G such that $g_i \notin I[p_G(S) - g_i]$, for $i = 1, 2$, and vertices h_1, h_2 in H such that $h_i \notin I[p_H(S) - h_i]$, for $i = 1, 2$. At least one of the four vertices of $\{(g_i, h_j) : i, j \in \{1, 2\}\}$, say (g_1, h_1) , is not in S , as otherwise $|p_G(S)| \cdot |p_H(S)| = 4$. Hence, by Lemma 4, $(g_1, h_1) \notin I[S]$ (see Figure 4(d)). \square

Proposition 8. Let $n \geq 5$ be an odd integer. If $m \geq 2$, then $g(K_m \boxtimes C_n) = 5$ and $h(K_m \boxtimes C_n) = 3$.

Proof. Notice that if $n = 2k + 1$, then the set $S = \{0, 1, k, k + 1, k + 2\}$ is a geodetic set of C_n satisfying condition (A), which means that $g(K_m \boxtimes C_n) \leq 5$. The equality is directly derived from Proposition 7, since every set of vertices of K_m trivially satisfies condition (B).

To prove that $h(K_m \boxtimes C_n) = 3$ it suffices to see that $h(K_m \boxtimes C_n) > 2$, as according to Theorem 3, $h(K_m \boxtimes C_n) \leq h(C_n) = 3$. To this end, take an arbitrary set $R = \{(i_1, j_1), (i_2, j_2)\}$ of cardinality 2 in $K_m \boxtimes C_n$. If $j_1 = j_2$, then $CH(R) = R$, i.e., in this case R is not a hull set of $h(K_m \boxtimes C_n)$. Assume thus that $j_1 \neq j_2$, and wlog that $R = \{(0, 0), (i, h)\}$, where $i \in \{0, 1\}$, $n = 2k + 1$ and $0 < h \leq k$. Certainly, $CH(R) \setminus R = \bigcup_{j=1}^{h-1} (K_m \times \{j\})$, i.e., neither in this case R is a hull set of $h(K_m \boxtimes C_n)$. \square

Proposition 9. For any $m, k \geq 2$, $5 \leq g(P_m \boxtimes C_{2k+1}) \leq 6$.

Proof. As an immediate consequence of Proposition 3 and Theorem 1 we obtain that $4 \leq g(P_m \boxtimes C_n) \leq 6$. Moreover, observe that every set $S \subset V(P_m)$ such that $2 \leq |S| \leq 4$ satisfies condition (B), which according to Proposition 7, allows us to derive that $g(P_m \boxtimes C_{2k+1}) \geq 5$. \square

Let us remark that both bounds are sharp, since it is straightforward to check that $g(P_3 \boxtimes C_7) = 5$ and $g(P_3 \boxtimes C_5) = 6$.

Proposition 10. For any $k, m \geq 2$, $h(C_{2k+1} \boxtimes P_m) = \begin{cases} 2, & \text{if } k \leq m-2; \\ 3, & \text{if } k \geq m-1. \end{cases}$

Proof. Certainly, $2 \leq h(C_{2k+1} \boxtimes P_m) \leq 3$, being the upper bound a corollary of Theorem 3, whereas the lower bound is an immediate consequence of the fact that $h(G) = 1$ if and only if $G = K_1$.

By symmetry reasons, we label the vertex set of C_{2k+1} as follows: $V(C_{2k+1}) = \Lambda = \{-k, \dots, -1, 0, 1, \dots, k\}$, whereas the vertex set labeling is the usual one: $V(P_m) = \Pi = \{0, 1, \dots, m-1\}$. In addition, we identify $V(C_{2k+1} \boxtimes P_m)$ with the grid $\mathcal{P} = \Lambda \times \Pi$ of points of the discrete plane \mathbb{Z}^2 .

Suppose first that $k \leq m-2$, m is odd and take the vertices $u = (0, 0)$ and $v = (0, m-1)$. Observe that, as shown in Figure 5 (a), $I[u, v]$ is the set of points belonging to the square determined by $\{u, v, x, y\}$, where $x = (-\frac{m-1}{2}, \frac{m-1}{2})$ and $y = (\frac{m-1}{2}, \frac{m-1}{2})$. Hence, since $\text{diam}(C_{2k+1}) = k < m-1$, we have that $\Lambda \times \{\frac{m-1}{2}\} \subseteq I^2[u, v]$. This fact together with Lemma 7, allows us to derive that $\{u, v\}$ is a hull set of $C_{2k+1} \boxtimes P_m$.

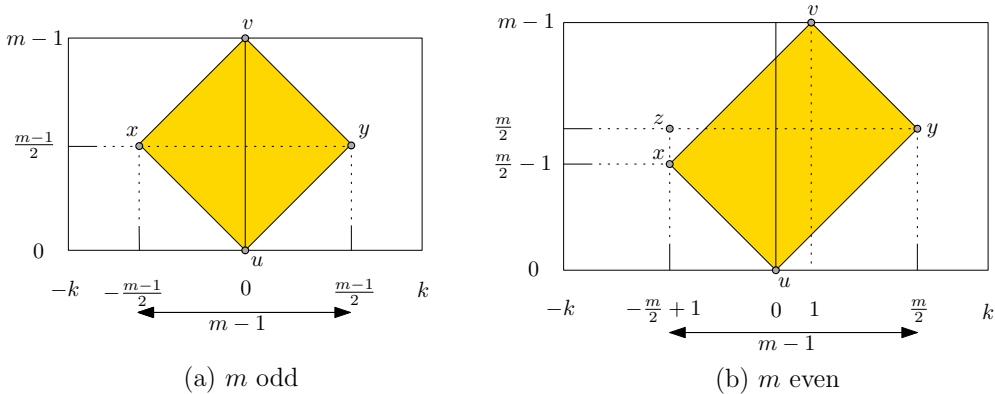


Figure 5: In both cases, $k \leq m-2$.

Next, assume that $k \leq m - 2$, m is even and take the pair of vertices $u = (0, 0)$ and $v = (1, m - 1)$. Notice that, as shown in Figure 5 (b), $I[u, v]$ is the set of points belonging to the rectangle determined by $\{u, v, x, y\}$, where $x = (-\frac{m}{2} + 1, \frac{m}{2} - 1)$ and $y = (\frac{m}{2}, \frac{m}{2})$. Hence, since $\text{diam}(C_{2k+1}) = k < m - 1 = \frac{m}{2} + (\frac{m}{2} - 1)$, we have that $\Lambda \times \{\frac{m}{2}\} \setminus \{z = (-\frac{m}{2} + 1, \frac{m}{2})\} \subseteq I^2[u, v]$, i.e., $\Lambda \times \{\frac{m}{2}\} \subseteq I^3[u, v]$. This fact together with Lemma 7, allows us to derive that $\{u, v\}$ is a hull set of $C_{2k+1} \boxtimes P_m$.

Finally, assume that $k \geq m - 1$ and take an arbitrary 2-vertex set $\{u, v\} \subset V(C_{2k+1} \boxtimes C_{2k+1})$. We may assume wlog that $u = (0, h)$ and $v = (a, h')$, where $0 \leq a \leq k$ and $0 \leq h \leq h'$. We distinguish two cases.

Case 1. $d(u, v) = \max\{a, h' - h\} = a > 0$: The path ρ of C_{2k+1} whose vertex set is $V(\rho) = \{0, 1, 2, \dots, a\}$ is the unique $0 - a$ geodesic. Hence, according to Lemma 3, $I[u, v]$ is the subset of points of \mathcal{P} lying in the rectangle \mathcal{R} determined by the four lines passing through either u or v , of slopes ± 1 . Note that, as shown in Figure 6 (a), this rectangle is inside the square of side length a determined by the four vertices of \mathcal{R} . This fact, together with Lemma 3, implies that for any pair of vertices $u', v' \in I[u, v] = \mathcal{R}$, the set $I[u', v']$ is the rectangle \mathcal{R}' contained in \mathcal{R} , determined by the four lines passing through either u' or v' , of slopes ± 1 . This means that $I[u, v]$ is a proper convex subset of $C_{2k+1} \boxtimes P_m$, and thus $\{u, v\}$ is not geodesic.

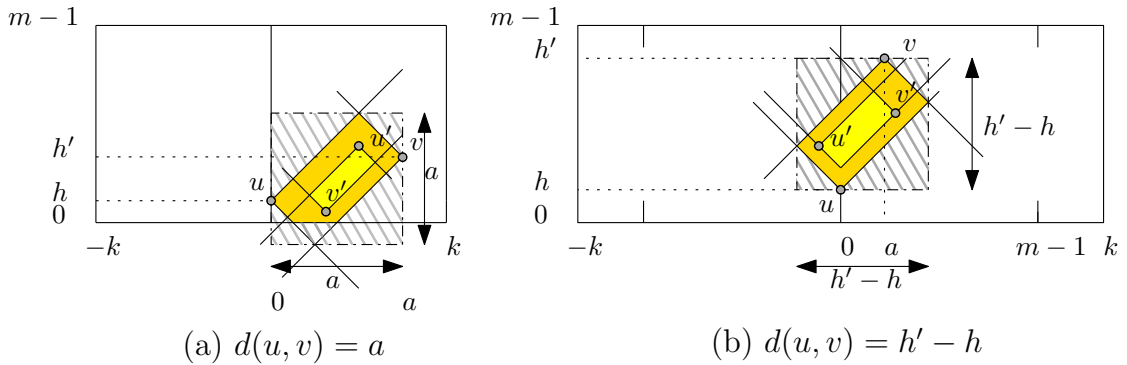


Figure 6: In both cases, $\{u = (0, h), v = (a, h')\} \subset V(C_{2k+1} \boxtimes P_m)$.

Case 2. $d(u, v) = \max\{a, h' - h\} = h' - h > 0$: As shown in Figure 6 (b), $I[u, v]$ consists of all points of \mathcal{P} lying in the rectangle \mathcal{R} determined by the four lines passing through either u or v , of slopes ± 1 . And it is inside the square of side length $h' - h \leq m - 1 \leq k$ determined by the four

vertices of \mathcal{R} . Hence, reasoning as in the preceding case and having in mind that $h' - h \leq k$, we derive that $I[u, v]$ is a proper convex subset of $C_{2k+1} \boxtimes P_m$, and thus $\{u, v\}$ is not geodetic. \square

The last strong product graphs we have studied is the so-called family of strong toruses, i.e., the strong product of two cycles. As an immediate consequence of Proposition 3 and Theorem 1, the following results are derived.

Proposition 11. *Let m, n be two integers such that $4 \leq \min\{m, n\}$.*

(i) *if both m and n are even, then $g(C_m \boxtimes C_n) = 4$*

(ii) *if mn is even, then $4 \leq g(C_m \boxtimes C_n) \leq 6$.*

In addition, we have been able to obtain a number of further results for the geodetic number, involving odd cycles.

Proposition 12. *Let h, k, n be integers such that $2 \leq \min\{h, k\}$ and $4 \leq n$.*

(i) *If $2h \leq k$, then $g(C_{2h+1} \boxtimes C_{2k}) = 4$.*

(ii) *If $5(2h - 1) \leq n$, then $g(C_{2h+1} \boxtimes C_n) \leq 5$.*

(iii) *If $2h + 1 \leq k$, then $g(C_{2h+1} \boxtimes C_{2k+1}) \leq 6$.*

(iv) *If $3 \leq h \leq k$, then $5 \leq g(C_{2h+1} \boxtimes C_{2k+1}) \leq 7$.*

Proof. (i) It is straightforward to verify that $V(C_{2h+1} \boxtimes C_{2k}) = I[S_1] = I[u_1, u_3] \cup I[u_2, u_4]$, where $S_1 = \{u_1, u_2, u_3, u_4\}$, $u_1 = (0, 0)$, $u_2 = (h, h)$, $u_3 = (0, k)$ and $u_4 = (h, h + k - 1)$.

(ii) It is straightforward to verify that $V(C_{2h+1} \boxtimes C_n) = I[S_2] = I[w_1, w_3] \cup I[w_1, w_4] \cup I[w_2, w_4] \cup I[w_2, w_5] \cup I[w_3, w_5]$, where $S_2 = \{w_1, w_2, w_3, w_4, w_5\}$, $t = \lfloor \frac{n}{5} \rfloor$, $w_1 = (h, 0)$, $w_2 = (h, t)$, $w_3 = (h, 2t)$, $w_4 = (h, 3t)$ and $w_5 = (h, 4t)$.

(iii) It is straightforward to verify that $V(C_{2h+1} \boxtimes C_{2k+1}) = I[S_3] = I[u_1, u_3] \cup I[u_2, u_4] \cup I[u_2, u_6] \cup I[u_3, u_5]$, where $S_3 = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $u_5 = (0, 2k)$ and $u_6 = (h, h + k)$.

(iv) The lower bound is a direct consequence of Lemma 9 and Proposition 7. The upper bound is obtained as a consequence of the following claim: $V(C_{2h+1} \boxtimes C_{2k+1}) = I[S_4] = I[v_1, v_2] \cup$

$I[v_1, v_5] \cup I[v_1, v_7] \cup I[v_2, v_3] \cup I[v_2, v_4] \cup I[v_2, v_6] \cup I[v_3, v_4] \cup I[v_3, v_5] \cup I[v_4, v_5] \cup I[v_5, v_6] \cup I[v_5, v_7] \cup I[v_6, v_7]$, where $S_4 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $v_1 = (0, 0)$, $v_2 = (1, k)$, $v_3 = (2, 2k)$, $v_4 = (h, k-1)$, $v_5 = (h+1, 2k-1)$, $v_6 = (h+2, k-2)$ and $v_7 = (2h, 2k-2)$.

To prove this claim, let us first partition the vertex set of $C_{2h+1} \boxtimes C_{2k+1}$ as shown in Figure 7, according to the following facts:

$$\begin{aligned}
(1) \quad 0 \leq i+j \leq h-1 \text{ and } & \begin{cases} (1.1) & i-j \leq 0 & \Rightarrow & (i, j) \in I[v_1, v_2] \\ (1.2) & 0 < i-j \leq 3 & \Rightarrow & (i, j) \in I[v_3, v_4] \\ (1.3) & 3 < i-j & \Rightarrow & (i, j) \in I[v_3, v_5] \end{cases} \\
(2) \quad h-1 < i+j < k+1 \text{ and } & \begin{cases} (2.0) & i=0, j=k & \Rightarrow & (i, j) \in I[v_2, v_6] \\ (2.1) & 1-k \leq i-j \leq 0 & \Rightarrow & (i, j) \in I[v_1, v_2] \\ (2.2) & 0 < i-j \leq 3 & \Rightarrow & (i, j) \in I[v_3, v_4] \\ (2.3) & 3 < i-j \leq h+3 & \Rightarrow & (i, j) \in I[v_3, v_5] \\ (2.4) & h+3 < i-j & \Rightarrow & (i, j) \in I[v_1, v_5] \end{cases} \\
(3) \quad k+1 \leq i+j \leq h+k-1 \text{ and } & \begin{cases} (3.1) & i-j \leq 1-k & \Rightarrow & (i, j) \in I[v_2, v_3] \\ (3.2) & 1-k < i-j \leq 1+h-k & \Rightarrow & (i, j) \in I[v_2, v_4] \\ (3.3) & 1+h-k < i-j < 4+h-k & \Rightarrow & (i, j) \in I[v_3, v_4] \\ (3.4) & 4+h-k \leq i-j \leq h+3 & \Rightarrow & (i, j) \in I[v_5, v_6] \\ (3.5) & h+3 < i-j & \Rightarrow & (i, j) \in I[v_1, v_2] \cup I[v_1, v_5] \end{cases} \\
(4) \quad h+k-1 < i+j < 2k+2 \text{ and } & \begin{cases} (4.1) & i-j < 2-2k & \Rightarrow & (i, j) \in I[v_1, v_7] \\ (4.2) & 2-2k \leq i-j < 1-k & \Rightarrow & (i, j) \in I[v_2, v_3] \\ (4.3) & 1-k \leq i-j \leq 1+h-k & \Rightarrow & (i, j) \in I[v_4, v_5] \\ (4.4) & 1+h-k < i-j \leq 4+h-k & \Rightarrow & (i, j) \in I[v_6, v_7] \\ (4.5) & 4+h-k < i-j \leq h+3 & \Rightarrow & (i, j) \in I[v_2, v_6] \\ (4.6) & h+3 < i-j & \Rightarrow & (i, j) \in I[v_1, v_5] \end{cases} \\
(5) \quad 2k+2 \leq i+j \leq h+2k \text{ and } & \begin{cases} (5.1) & i-j \leq 2+h-2k & \Rightarrow & (i, j) \in I[v_3, v_5] \\ (5.2) & 2+h-2k < i-j \leq 1+h-k & \Rightarrow & (i, j) \in I[v_4, v_5] \\ (5.3) & 1+h-k < i-j \leq 4+h-k & \Rightarrow & (i, j) \in I[v_6, v_7] \\ (5.4) & 4+h-k < i-j \leq h+3 & \Rightarrow & (i, j) \in I[v_2, v_6] \\ (5.5) & h+3 < i-j & \Rightarrow & (i, j) \in I[v_1, v_2] \end{cases} \\
(6) \quad h+2k \leq i+j \leq 2h+2k-2 \text{ and } & \begin{cases} (6.0) & i=h+1, j=2k & \Rightarrow & (i, j) \in I[v_5, v_6] \\ (6.1) & 2+h-2k \leq i-j \leq 2+2h-2k & \Rightarrow & (i, j) \in I[v_5, v_7] \\ (6.2) & 2+2h-2k < i-j \leq 4+h-k & \Rightarrow & (i, j) \in I[v_6, v_7] \\ (6.3) & 4+h-k < i-j & \Rightarrow & (i, j) \in I[v_2, v_6] \end{cases} \\
(7) \quad 2h+2k-2 < i+j \leq 2h+2k & \Rightarrow (i, j) \in I[v_1, v_7]
\end{aligned}$$

It is straightforward to see that each of the regions described above is contained in at least one of the 12 mentioned closed intervals. As a matter of example, notice that if $h = k$, then regions (1.1), (2.1) and (5.5) are completely covered by $I[v_1, v_2]$, since:

$$(i, j) \in I[v_1, v_2] \Leftrightarrow \begin{cases} 0 \leq i \leq 2h \\ 0 \leq j \leq 2k \end{cases} \text{ and either } \begin{cases} 1 - k \leq i - j \leq 0 \\ 0 \leq i + j \leq k + 1 \end{cases} \text{ or } \begin{cases} 2h - k + 2 \leq i - j \\ 2h + 1 \leq i + j \end{cases} \quad \square$$

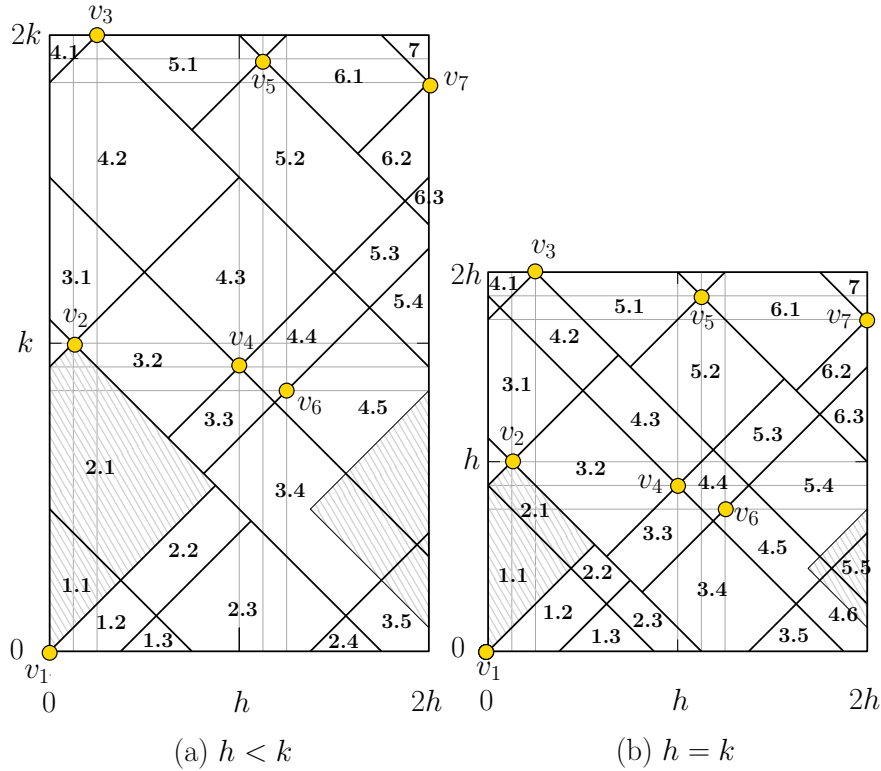


Figure 7: Partition of $V(C_{2h+1} \boxtimes C_{2k+1})$. The dashed region is $I[v_1, v_2]$.

Let us remark that all bounds presented in the last two propositions can not be improved, as it is shown in Table 3, which contains the geodetic number of some strong product graphs of the form $C_5 \boxtimes C_n$ computationally obtained.

| $C_5 \boxtimes C_n$ | $C_5 \boxtimes C_4$ | $C_5 \boxtimes C_5$ | $C_5 \boxtimes C_6$ | $C_5 \boxtimes C_7$ | $C_5 \boxtimes C_8$ | $C_5 \boxtimes C_9$ |
|------------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| $g(C_5 \boxtimes C_n)$ | 5 | 5 | 6 | 7 | 4 | 6 |

Table 3: Geodetic number of some strong product graphs of the form $C_5 \boxtimes C_n$.

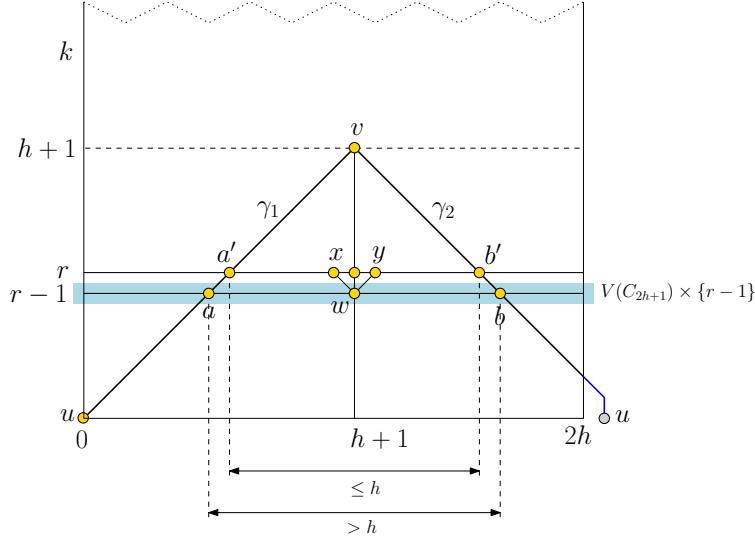


Figure 8: γ_1 and γ_2 are two $u-v$ geodesics in $C_{2h+1} \boxtimes C_{2k+1}$, whenever $h < k$.

Proposition 13. *If h, k are integers such that $2 \leq h \leq k$, then $h(C_{2h+1} \boxtimes C_{2k+1}) = \begin{cases} 2, & \text{if } h < k; \\ 3, & \text{if } h = k. \end{cases}$*

Proof. Certainly, $2 \leq h(C_{2k+1} \boxtimes C_{2k+1}) \leq 3$, being the upper bound a corollary of Theorem 3, whereas the lower bound is derived from the fact that $h(G) = 1$ if and only if $G = K_1$.

Suppose next that $h < k$ and consider the set $S = \{u, v\}$, where $u = (0, 0)$ and $v = (h+1, h+1)$ (see Figure 8). Since $d(u, v) = h+1$, we have that both $\gamma_1 : (0, 0)(1, 1) \dots (h, h)(h+1, h+1)$ and $\gamma_2 : (0, 0)(0, 1)(2h, 2) \dots (h+2, h)(h+1, h+1)$ are $u-v$ geodesic. Observe that given $c = (i, i) \in V(\gamma_1)$ and $d = (2h+2-i, i) \in V(\gamma_2)$, $d(c, d) = 2h+2-2i$ if and only if $2h+2-2i \leq h$, i.e., if and only if $\lceil \frac{h+2}{2} \rceil \leq i$. At this point, we claim that if $r = \lceil \frac{h+2}{2} \rceil$, then $V(C_{2h+1}) \times \{r-1\} \subseteq I^4[S]$, which according to Lemma 7, is enough to end the proof of the case $h < k$.

To show that this claim is true, consider the vertices $a = (r-1, r-1)$, $w = (h+1, r-1)$, $b = (2h-r+3, r-1)$, $a' = (r, r)$, $x = (h, r)$, $y = (h+2, r)$, $b' = (2h-r+2, r)$ and observe:

- * $\{a, b, a', b'\} \subset I[S]$, since $\{a, a'\} \subset V(\gamma_1)$ and $\{b, b'\} \subset V(\gamma_2)$.
- * $\{x, y\} \subset I[a', b'] \subseteq I^2[S]$, since $r < h < h+2 < 2h-r+2$ and $d(a', b') = 2h-2r+2 \leq h$.
- * $w \in I[x, y] \subseteq I^3[S]$, since $d(x, w) = d(w, y) = 1$ and $d(x, y) = 2$.

Finally, we show that for every $i \in V(C_{2h+1})$, the vertex $z = (i, r-1) \in I^4[S]$:

- If $0 \leq i \leq r-1$, then $z \in I[a, b] \subset I^2[S]$, since $(2h-r+3) - (r-1) = 2h-2r+4 \geq h+1$.
- If $r-1 \leq i \leq h+1$, then $z \in I[a, w] \subset I^4[S]$, since $(h+1) - (r-1) = h-r+2 \leq h$.
- If $h+1 \leq i \leq 2h-r+3$, then $z \in I[w, b] \subset I^4[S]$, since $(2h-r+3) - (h+1) = h-r+2 \leq h$.
- If $2h-r+3 \leq i \leq 2h$, then $z \in I[a, b] \subset I^2[S]$, since $(2h-r+3) - (r-1) \geq h+1$.

Now, suppose that $h = k$ and take an arbitrary 2-vertex set $\{u, v\} \subset V(C_{2k+1} \boxtimes C_{2k+1})$. By symmetry reasons, we label the vertex set of C_{2k+1} as follows: $V(C_{2k+1}) = \Lambda = \{-k, \dots, -1, 0, 1, \dots, k\}$.

We may assume wlog that $u = (0, 0)$ and $v = (a, b)$, where $0 \leq b \leq a \leq k$. Observe that $d(u, v) = \max\{a, b\} = a \leq k$, and that the path ρ of C_{2k+1} whose vertex set is $V(\rho) = \{0, 1, 2, \dots, a\}$ is the unique $0 - a$ geodesic. Hence, according to Lemma 3,

$$I[u, v] = \{(i, j) : 0 \leq i \leq a, |j| \leq i, |j - b| \leq a - i\} = \{(i, j) : 0 \leq i - j \leq a - b, 0 \leq i + j \leq a + b\}$$

In other words, if we identify $V(C_{2k+1} \boxtimes C_{2k+1})$ with the grid $\mathcal{P} = \Lambda \times \Lambda$ of points of the discrete plane \mathbb{Z}^2 , then $I[u, v]$ is the subset of points of \mathcal{P} lying in the rectangle \mathcal{R} determined by the four lines passing through either u or v , of slopes ± 1 . Note that, as shown in Figure 9, this rectangle is inside the square of side length a determined by the four vertices of \mathcal{R} . This fact, together with Lemma 3, implies that for any pair of vertices $u', v' \in I[u, v] = \mathcal{R}$, the set $I[u', v']$ is the rectangle \mathcal{R}' contained in \mathcal{R} , determined by the four lines passing through either u' or v' , of slopes ± 1 . This means that $I[u, v]$ is a proper convex subset of $C_{2k+1} \boxtimes C_{2k+1}$, and thus $\{u, v\}$ is not geodetic. \square

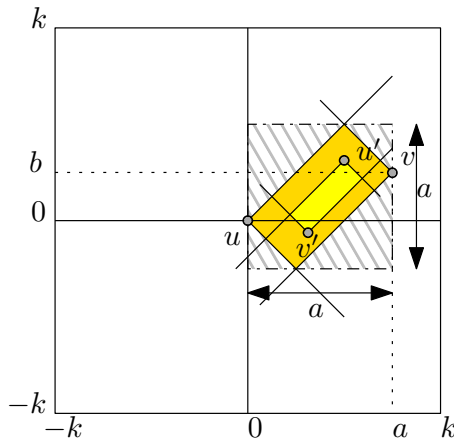


Figure 9: $u = (0, 0)$ and $v = (a, b)$ are two vertices of the strong torus $C_{2k+1} \boxtimes C_{2k+1}$

We conclude this section by showing in Table 4 all the results obtained for strong product graphs of the form $G \boxtimes C_n$.

| $G \boxtimes C_n$ | $g(G \boxtimes C_n)$ | $h(G \boxtimes C_n)$ |
|---------------------|---|--|
| $P_m \boxtimes C_n$ | $\begin{cases} 4, & \text{if } n \text{ is even;} \\ 5, 6, & \text{if } n \text{ is odd.} \end{cases}$ | $\begin{cases} 3, & \text{if } n = 2r + 1 \text{ odd and } m < r + 2; \\ 2, & \text{otherwise.} \end{cases}$ |
| $K_m \boxtimes C_n$ | $\begin{cases} 4, & \text{if } n \text{ is even;} \\ 5, & \text{if } n \text{ is odd.} \end{cases}$ | $\begin{cases} 2, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$ |
| $C_m \boxtimes C_n$ | $\begin{cases} 4, & \text{if } m \text{ and } n \text{ are even;} \\ 4 - 6, & \text{if } m \text{ is even and } n \text{ is odd;} \\ 5 - 7, & \text{if } m \text{ and } n \text{ are odd.} \end{cases}$ | $\begin{cases} 3, & \text{if } m = n \text{ is odd;} \\ 2, & \text{otherwise.} \end{cases}$ |

Table 4: Geodetic and hull numbers of some strong product graphs of the form $G \boxtimes C_n$.

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